

Dynamical system approach to scalar-vector-tensor cosmology

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Abstract

Using scalar-vector-tensor Brans Dicke (VBD) gravity [3] in presence of self interaction BD potential $V(\phi)$ and perfect fluid matter field action we solve corresponding field equations via dynamical system approach for flat Friedmann Robertson Walker metric (FRW). We obtained 3 type critical points for Λ CDM vacuum de Sitter era where stability of our solutions are depended to choose particular values of BD parameter ω . One of these fixed points is supported by a constant potential which is stable for $\omega < 0$ and behaves as saddle (quasi stable) for $\omega \geq 0$. Two other ones are supported by a linear potential $V(\phi) \sim \phi$ which one of them is stable for $\omega = 0.27647$. For a fixed value of ω there is at least 2 out of 3 critical points reaching to a unique critical point. Namely for $\omega = -0.16856(-0.56038)$ the second (third) critical point become unique with the first critical point. In dust and radiation eras we obtained 1 critical point which never become unique fixed point. In the latter case coordinates of fixed points are also depended to ω . To determine stability of our solutions we calculate eigenvalues of Jacobi matrix of 4D phase space dynamical field equations for de Sitter, dust and radiation eras. We should be point also potentials which support dust and radiation eras must be similar to $V(\phi) \sim \phi^{-\frac{1}{2}}$ and $V(\phi) \sim \phi^{-1}$ respectively. In short our study predicts that radiation and dust eras of our VBD-FRW cosmology transmit to stable de Sitter state via non-constant potential (effective variable cosmological parameter) by choosing $\omega = 0.27647$.

1 Introduction

The physical nature itself is complex system really and is described by nonlinear chaotic dynamics [4,5,6]. A chaotic dynamics in its continues (discrete) form is described by nonlinear (iteration maps) differential equations. It leads usually to its possible stable points called as attractors (see arrow diagrams

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in figure 1). Usually dynamical systems described by nonlinear differential equations have not regular analytic solutions. This restrict us to choose geometrical approach to solve them. The latter method gives us properties of the solutions without the solutions themselves. Properties of the solutions are called as attractors (sink and/or stable) and saddles (quasi-stable). The phase space variables of dynamical system at the classical mechanics are well known as canonical coordinates and corresponding momenta but not in the cosmological context. In the latter case dynamical variables are more and there are several degrees of freedom to choose them. If we choose unsuitable choices so can not obtain physically applicable solutions according to the experimental context. This restrict us to regard two important statements about the geometrical variables as must be dimensionless and bounded. The latter two properties make as finite the phase space which means all of the critical points become visible.

Choosing some suitable dimensionless geometrical variables one can reduces a ‘ n ’ order nonlinear differential equation of a dynamical system to number of ‘ n ’ to first order differential equations as

$$\vec{\dot{x}} = \frac{d\vec{x}}{dt} = f(\vec{x}, t) \quad (1.1)$$

where $\vec{x} = \{x^i; i = 1, 2, 3, \dots, n\} \in E \subseteq R^n$ is state of ‘ n ’-dimensional phase space $E \subseteq R^n$. The equation (1.1) is called as autonomous if

$$\frac{\partial f}{\partial t} = 0 \quad (1.2)$$

and non-autonomous if

$$\frac{\partial f}{\partial t} \neq 0. \quad (1.3)$$

Solutions of the equation $\vec{\dot{x}} = 0$ gives us critical points P_c of the dynamical system. One can obtain eigenvalues λ_i of each critical point by calculating Jacobi matrix of the vector function $f(\vec{x}, t)$ defined by

$$[\mathcal{J}]_{ij} = J_{ij} = \left(\frac{\partial \dot{x}^i}{\partial x^j} \right)_{x=x_{critical}} \quad (1.4)$$

and solving its secular equation as

$$\det(J_{ij} - \lambda \delta_{ij}) = 0. \quad (1.5)$$

This leads to an algebraic equation which its solutions give us eigenvalues of a critical point of the dynamical system. Characters of obtained critical points are depended to numerical value of the corresponding eigenvalues. For instance, the critical point is called as unstable (repeller and/or source) if the corresponding eigenvalues take positive real value numerically. If at least one of all real eigenvalues takes negative real value numerically then the critical point is called saddle. The critical points is called as stable (attractor and/or sink), if all of the eigenvalues take negative real value. If eigenvalues take complex numbers with positive (negative) real value then the critical point is called as spiral unstable (stable). For zero eigenvalues the system become degenerated and so we can not tell about stability and/or instability of the dynamical system under consideration (see table 1).

In context of cosmological models the dynamical system approach is used to obtain Λ CDM phase by more authors [7-17]: Zhou et al are used $f(G)$ gravity to study flat FRW cosmology in [7] where $G = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\lambda\eta}R_{\mu\nu\lambda\eta}$ is Gauss-Bonnet topological invariant. They are obtained two kinds of stable accelerated solutions called as de Sitter and phantom-like of dark energy regime. Azizi and Yaraie are used non-minimally matter coupled $f(R)$ gravity to study flat FRW cosmology in [8]. They are obtained vacuum de Sitter era of the Universe which can be mimic the late-time acceleration of the cosmic evolution. Hrycyna and Szydtowski are studied BD-FRW cosmology in presence of a quadratic scalar potential in [9,10] containing stable de Sitter phase and are studied observational constraints of the model in ref. [11]. Copeland et al analyzed the dynamics of a single scalar field in FRW universes with spatial curvature in ref. [12] where an attractor critical point is obtained to satisfy de Sitter and power law expanding Universe. Matos et al studied dynamical approach of scalar-tensor cosmology in presence of a cosh type of the potential plus a cosmological constant reducing to de Sitter attractor [13]. Lopez and Ibarra studied attractor properties of the chaotic inflation in [14] by using a minimally coupled scalar field in presence of a quadratic scalar potential. Amendola was used quintessence (light scalar field) effects to study cosmic acceleration via dynamical system approach in presence of dynamical cosmological parameter and an exponential potential [15]. He obtained a multi-pole spectrum effect of the microwave background at large angles where the acoustic peaks are shifted and their amplitude is changed. Fay et al are obtained particular class of $f(R)$ modified gravity theories which can be mimic Λ CDM cosmology in ref. [16]. Nozari and Kiani are studied (1+4) dimensional bran-world cosmology containing a Gauss-Bonnet

term at bulk action and obtained stable de Sitter phase state [17]. In short, we know that Λ CDM phase of accelerating Universe is supported via ansatz of unknown cosmological constant Λ in general theory of relativity and dark matter inflaton scalar field in more scalar tensor gravity theories. Really, origin of the non-baryonic dark matter proposal is not known and there are more candidate for it [18,19]. Diversity of dark matter particles candidate encouraged more authors to present alternative models as scalar-vector-tensor gravity theories (TeVeS) without non-baryonic dark matter which one can use instead of the general theory of relativity itself and/or usual scalar tensor models. In these models dynamical vector fields are four velocity of preferred reference frames satisfying general covariance condition. These vector fields support acceleration of the expanding Universe, galaxy rotation curves and corrections on gravitational acceleration law in solar system, astrophysical and cosmological scales [20] instead of less-known non-baryonic dark matter (see also refs. [21,22] for their experimental constraints). If dynamical vector fields to be have unit-time-like property then the scalar-vector-tensor gravities can be also support metric signature transition dynamics from Euclidean $(+,+,+,+)$ to Lorentzian $(-,+,+,+)$ signature (see [3] and reference therein). In short, the model presented in ref. [3] is generalized BD gravity [2] by transforming the background metric $g_{\mu\nu}$ to $g_{\mu\nu} + 2N_\mu N_\nu$. Flat FRW quantum cosmology of the model and its metric signature transition property were studied in refs. [23] and [24] respectively.

In this paper we use dynamical system approach of the gravity model [3] in presence of self-interacting BD potential and matter-radiation perfect fluid cosmic source and obtain Λ CDM de Sitter stable phase of the accelerating flat FRW Universe. Originally, vector field stress tensor of our used model which support inflation of the cosmological Universe makes free of Jordan and/or Einstein frame of the used BD gravity. While Salcedo et al is shown in ref. [1] that the scalar tensor Brans Dicke (SBD) gravity itself [2] in presence of quadratic self-interaction potential their attractor de Sitter solution is only valid for Jordan frame. Hence they claimed that the BD gravity itself dose have not a Λ CDM phase as an universal attractor. Form the latter view our work can be outstanding and so considerable to study with more details. As an experimental result they obtained time variation of the Newton's gravitational coupling parameter as $|\dot{G}/G| < 9 \times 10^{-13} \text{ yr}^{-1}$ for experimental values of Hubble constant $H_0 = 7.24 \times 10^{-11} \text{ yr}^{-1}$ and BD parameter $\omega = 40000$ while we obtained its corrections coming from preferred reference frame effects (dynamical vector fields corrections). Organization of

the work is as follows.

In section 2 we call the gravity model [3] and calculate its dynamical field equations. In section 3 we obtain Friedmann equations of the model. Next we make 4D cosmic dynamical phase space to write corresponding dynamical equations. Then we obtain critical points, matrix Jacobi and their eigenvalues for Λ CDM vacuum de Sitter, dust and radiation eras. Finally we denote to concluding remark in section 4.

2 The Model

Let us we start with the following scalar-vector-tensor-gravity action [3].

$$I_{total} = I_{BD} + I_N + I_m + I_r \quad (2.1)$$

which with assumption $\epsilon = 0$ the term

$$I_{BD} = \frac{1}{16\pi} \int dx^4 \sqrt{g} \left\{ \phi R - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right\} \quad (2.2)$$

is BD scalar tensor gravity itself [2]. $V(\phi)$ is called as BD self interaction potential and

$$I_N = \frac{1}{16\pi} \int dx^4 \sqrt{g} \{ \zeta(x^\nu) (g^{\mu\nu} N_\mu N_\nu + 1) + 2\phi F_{\mu\nu} F^{\mu\nu} - \phi N_\mu N^\nu (2F^{\mu\lambda} \Omega_{\nu\lambda} + F^{\mu\lambda} F_{\nu\lambda} + \Omega^{\mu\lambda} \Omega_{\nu\lambda} - 2R^\nu_\mu + \frac{2\omega}{\phi^2} \nabla_\mu \phi \nabla^\nu \phi) \}, \quad (2.3)$$

with

$$F_{\mu\nu} = 2(\nabla_\mu N_\nu - \nabla_\nu N_\mu), \quad \Omega_{\mu\nu} = 2(\nabla_\mu N_\nu + \nabla_\nu N_\mu) \quad (2.4)$$

describes action of unit time like dynamical four velocity $N_\mu(x^\nu)$ of a preferred reference frame. Up to $\zeta(x^\nu)$ term which is used as ansatz, the action (2.3) is obtained by transforming metric field of the BD action (2.2) as $g_{\mu\nu} \rightarrow g_{\mu\nu} + 2N_\mu N_\nu$. Details of calculations are given in ref. [3]. Matter and radiation counterparts of a perfect fluid source is considered as

$$I_m = \frac{1}{16\pi} \int dx^4 \sqrt{g} L_m \quad (2.5)$$

and

$$I_r = \frac{1}{16\pi} \int dx^4 \sqrt{g} L_r \quad (2.6)$$

where L_m and L_r are the matter and radiation lagrangian densities respectively. $T_{\mu\nu}$ is the matter-radiation stress energy-momentum tensor and is given against the corresponding Lagrangian density as follows.

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}(L_m + L_r))}{\delta g^{\mu\nu}}. \quad (2.7)$$

We will assume $T_{\mu\nu}$ to be stress energy-momentum tensor of a perfect fluid in what follows as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \quad (2.8)$$

Here u^μ is time-like four velocity of the fluid satisfying $g^{\mu\nu}u_\mu u_\nu = -1$. $\rho = \rho_m + \rho_r$ and $p = p_m + p_r$ where $\rho_m(\rho_r)$ is matter (radiation) counterpart energy density of the fluid and $p_m(p_r)$ is corresponding isotropic hydrostatic pressure. The action (2.3) shows that the vector field N_μ is coupled as non-minimally with the BD scalar field ϕ . The action (2.1) is written in units $c = \hbar = 1$ with Lorentzian signature $(-, +, +, +)$. The undetermined Lagrange multiplier $\zeta(x^\nu)$ controls that N_μ to be an unit time-like vector field. ϕ describes inverse of variable Newton's gravitational coupling parameter and its dimension is $(length)^{-2}$ in units $c = \hbar = 1$. Absolute value of determinant of the metric $g_{\mu\nu}$ is defined by g . Present limits of dimensionless BD parameter ω based on time-delay experiments [25,26,27,28] requires $\omega \geq 4 \times 10^4$. General relativistic approach of the BD gravity action (2.2) is obtained by setting $V(\phi) = 0$ and $\omega \rightarrow \infty$. Varying (2.1) with respect to $\zeta(x^\nu)$, ϕ , N^μ and $g^{\mu\nu}$ we obtain respectively

$$g^{\mu\nu}N_\mu N_\nu = -1, \quad (2.9)$$

$$\begin{aligned} & \frac{2\omega\Box\phi}{\phi} - \frac{\omega g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{\phi^2} - \frac{4\omega N^\mu N^\nu\partial_\mu(\sqrt{g}\partial_\nu\phi)}{\phi\sqrt{g}} - \frac{dV(\phi)}{d\phi} - \frac{4\omega\partial_\mu(N^\mu N^\nu)\partial_\nu\phi}{\phi} \\ & - \frac{4\omega\Gamma_{\mu\alpha}^\mu N^\alpha N^\nu\partial_\nu\phi}{\phi} - \frac{4\omega\Gamma_{\mu\lambda}^\nu N^\mu N^\lambda\partial_\nu\phi}{\phi} + \frac{2\omega N^\mu N^\nu\partial_\mu\phi\partial_\nu\phi}{\phi^2} + R - 2N^\mu N^\nu R_{\mu\nu} \\ & + 2F_{\mu\nu}F^{\mu\nu} - N_\mu N^\nu \{2F^{\mu\lambda}\Omega_{\nu\lambda} + F^{\mu\lambda}F_{\nu\lambda} + \Omega^{\mu\lambda}\Omega_{\nu\lambda}\} = 0, \quad (2.10) \\ & \frac{[4F_{\mu\nu} - N_\mu N^\lambda(F_{\lambda\nu} + 3\Omega_{\lambda\nu}) + N_\nu N^\lambda(F_{\lambda\mu} - \Omega_{\mu\lambda})]\partial^\mu(\sqrt{g}\phi)}{\sqrt{g}\phi} \\ & + \nabla^\mu[4F_{\mu\nu} - N_\mu N^\lambda(F_{\lambda\nu} + 3\Omega_{\lambda\nu}) + N_\nu N^\lambda(F_{\lambda\mu} - \Omega_{\mu\lambda})] \\ & + N_\mu(F_{\nu\lambda} + 3\Omega_{\nu\lambda})\nabla^\mu N^\lambda + N^\lambda(F_{\lambda\mu} + 3\Omega_{\lambda\mu})\nabla_\nu N^\mu - N_\lambda(F_{\nu\mu} - \Omega_{\mu\nu})\nabla^\mu N^\lambda \end{aligned}$$

$$-N^\lambda(F_{\lambda\mu} - \Omega_{\mu\lambda})\nabla^\mu N_\nu + 2N^\mu R_{\mu\nu} - \frac{2\omega N^\mu \partial_\mu \phi \partial_\nu \phi}{\phi^2} - \frac{\zeta(x^\alpha)N_\nu}{\phi} = 0 \quad (2.11)$$

and

$$\begin{aligned} G_{\mu\nu} = & -\frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega \partial_\mu \phi \partial_\nu \phi}{\phi^2} + \frac{\partial_\mu(\sqrt{g}\partial_\nu \phi)}{\sqrt{g}\phi} - \frac{\zeta(x^\alpha)N_\mu N_\nu}{\phi} + \frac{2\Box(\phi N_\mu N_\nu)}{\phi} \quad (2.12) \\ & -\frac{g_{\mu\nu}}{2\phi}\left\{2\Box\phi + \frac{\omega g^{\alpha\beta}\partial_\alpha \phi \partial_\beta \phi}{\phi} - 2\phi F_{\alpha\beta}F^{\alpha\beta} + 2\phi N_\alpha N^\beta F^{\alpha\lambda}\Omega_{\beta\lambda}\right. \\ & \left.+ \phi N_\alpha N^\beta (F^{\alpha\lambda}F_{\beta\lambda} + \Omega^{\alpha\lambda}\Omega_{\beta\lambda}) + 2N^\alpha N^\beta (\phi R_{\alpha\beta} - \frac{\omega \partial_\alpha \phi \partial_\beta \phi}{\phi})\right\} + \frac{V(\phi)}{\phi}g_{\mu\nu}. \end{aligned}$$

where we defined

$$\Box = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu). \quad (2.13)$$

We now choose flat FRW background metric to study stability situations of Λ CDM vacuum de Sitter, dust and radiation eras of the model.

3 Cosmological setting

In context of homogenous and isotropic universes, one use usually FRW background metric which from point of view of a comoving observer, in flat case with Lorentizan signature $(-, +, +, +)$ is given by

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2). \quad (3.1)$$

$a(t)$ is scale factor of spatial part of the above metric. Applying (3.1) one can obtain a simple solution of the equation (2.9) described by Kronecker delta function as

$$N_\mu(t) = \delta_{t\mu} \quad (3.2)$$

where $N_\mu = 1$ for $\mu = t$ and $N_\mu = 0$ for $\mu \neq t$. Applying (3.2) and definition of covariant differentiation $\nabla \equiv \partial + \Gamma$ for (2.4) one can obtain

$$F_{\mu\nu}(t) = 0, \quad \Omega_{\mu\nu}(t) = 4a\dot{a} \text{diag}(0, 1, 1, 1) \quad (3.3)$$

where $\dot{} \equiv \frac{d}{dt}$. Applying (3.1), (3.2) and (3.3), the equations (2.10) and (2.11) become respectively

$$6\omega\dot{\psi} + 12\dot{H} + 30\omega H\psi + 5\omega\psi^2 + 18H^2 + \frac{dV(\phi)}{d\phi} = 0 \quad (3.4)$$

and

$$\frac{\zeta}{\phi} = 2\omega\psi^2 - 6(\dot{H} + H^2) \quad (3.5)$$

where we defined

$$\psi(t) = \frac{\dot{\phi}}{\phi}, \quad (3.6)$$

and

$$H(t) = \frac{\dot{a}}{a}. \quad (3.7)$$

Inserting (2.8), (3.1), (3.2), (3.3), (3.5), (3.6) and (3.7) one can obtain time-time and space-space components of the Einstein equation (2.12) respectively as follows.

$$G_t^t = 3H^2 = 8\pi\rho^*, \quad (3.8)$$

and

$$G_j^i = (2\dot{H} + 3H^2)\delta_j^i = -8\pi p^*\delta_j^i \quad (3.9)$$

where δ_j^i with $i, j \equiv \{x, y, z\}$ is 3 dimensional Keonecker delta function. Also we defined generalized fluid density ρ^* and corresponding isotropic pressure p^* as

$$8\pi\rho^* = \frac{8\pi(\rho_m + \rho_r)}{\phi} + \frac{(5\omega + 4)}{2}\psi^2 + 2\dot{\psi} - 9\dot{H} + 6H\psi - 9H^2 + \frac{V(\phi)}{\phi} \quad (3.10)$$

and

$$8\pi p^* = \frac{8\pi\rho_r}{3\phi} - \frac{(2 - \omega)\psi^2}{2} - \dot{\psi} + 3\dot{H} - 3H\psi + 3H^2 - \frac{V(\phi)}{\phi} \quad (3.11)$$

where $(\rho_m \neq 0, p_m = 0)$ and $(\rho_r, p_r) \neq 0$ with $p_r = \rho_r/3$ are matter and radiation components of the mixture perfect fluid with total density $\rho = \rho_m + \rho_r$ and pressure $p = p_r$. Applying (3.8) and (3.9), the Bianchi identity $\nabla_\mu G_\nu^\mu = 0$ leads to covariant conversation condition

$$\dot{\rho}^* + 3H(\rho^* + p^*) = 0. \quad (3.12)$$

Inserting (3.8), the above conservation condition can be rewritten as

$$\frac{p^*}{\rho^*} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} \quad (3.13)$$

which can be re-derived directly from (3.8) and (3.9). However one can eliminate \dot{H} term of the equation (3.5) by inserting (3.13) to obtain barotropic parameter of the effective fluid as

$$\gamma = \frac{p^*}{\rho^*} = -\frac{1}{3} - \frac{2\omega\psi^2}{9H^2} + \frac{\zeta}{9\phi H^2}. \quad (3.14)$$

The above equation shows that the fields $\zeta(t), \psi(t)$ and H can be control numerical values of γ . For instance $\psi = \zeta = 0$ reads to cosmic strings $\gamma = -\frac{1}{3}$. In what follows we will seek stability of phase solutions for de Sitter, dust and radiation eras by setting ansatz $\gamma = -1, 0, \frac{1}{3}$ respectively. One can obtain a good constraint condition between relative densities counterparts as

$$12(1+2\omega)\frac{V(\phi)}{\phi} + \frac{dV(\phi)}{d\phi} - 6(2+5\omega)\left(\frac{8\pi\rho_m}{\phi}\right) - 4(5+12\omega)\left(\frac{8\pi\rho_r}{\phi}\right) + 12\omega H\psi - \omega(43+102\omega)\psi^2 + 18(1+2\omega)H^2 = 0 \quad (3.15)$$

where matter and radiation densities counterparts $\rho_{m,r}$ satisfy separately the conservation equation respectively as follows.

$$\dot{\rho}_m + 3H\rho_m = 0 \quad (3.16)$$

and

$$\dot{\rho}_r + 4H\rho_r = 0. \quad (3.17)$$

The condition (3.15) is obtained from (3.11) when we eliminate $\dot{\psi}, p^*, \rho^*, \dot{H}$ via (3.4), (3.13), (3.8) and (3.5) respectively. Also we can obtain a suitable equation for ψ by applying (3.8), (3.9), (3.10) and (3.11) such that

$$\dot{\psi} = -(1+17\omega)\psi^2 + 6H^2 - 3H\psi - 8\left(\frac{8\pi\rho_r}{\phi}\right) - 5\left(\frac{8\pi\rho_m}{\phi}\right) + 4\frac{V(\phi)}{\phi}. \quad (3.18)$$

We now can solve the equations (3.14), (3.15), (3.16), (3.17) and (3.18) to determine the fields a, H, ϕ, ψ, ζ for given sources ρ_m, ρ_r , and $V(\phi)$ via dynamical system approach. To do so we must be first make 5 dimensionless phase space variables from a, H, ϕ, ψ, ζ and then obtain corresponding dynamical equations of phase space. To study stability of de Sitter epoch we must be evaluate critical points of phase space, and eigenvalues of corresponding Jacobi matrix as follows.

3.1 Cosmic dynamical system phase space

First we define dimensionless time derivative against e-folding parameter $\tau = \ln(a/a_i)$ of the expanding Universe as

$$' = \frac{d}{d\tau} = \frac{1}{H} \frac{d}{dt} \quad (3.19)$$

together with the following dimensionless variables of the cosmic phase space.

$$x(\tau) = \frac{\psi}{H}, \quad (3.20)$$

$$q(\tau) = \frac{\zeta}{\phi H^2}, \quad (3.21)$$

$$y(\tau) = \frac{8\pi\rho_m}{\phi H^2} \quad (3.22)$$

$$z(\tau) = \frac{8\pi\rho_r}{\phi H^2} \quad (3.23)$$

$$v(\tau) = \frac{V(\phi)}{\phi H^2} \quad (3.24)$$

and

$$s(\tau) = \frac{\dot{H}}{H^2}. \quad (3.25)$$

Inserting (3.19), (3.20), (3.21), (3.22), (3.23), (3.24), (3.25) into the equations (3.14), (3.15), (3.16), (3.17) and (3.18), one can obtain dimensionless dynamical equations of phase space variables as follows.

$$x' = -(1 + 17\omega)x^2 + \frac{3(\gamma - 1)}{2}x - 5y - 8z + 4v + 108 \quad (3.26)$$

$$y' = (3\gamma - 2 - x)y, \quad (3.27)$$

$$z' = [3(\gamma - 1) - x]z, \quad (3.28)$$

$$\begin{aligned} v' = & \omega(43 + 102\omega)x^3 - 12\omega x^2 + 18(1 + 2\omega)x + 6(2 + 5\omega)xy \\ & + 4(5 + 12\omega)xz - (13 + 24\omega)xv + \frac{3(1 + \gamma)v}{2} \end{aligned} \quad (3.29)$$

where

$$q = 2\omega x^2 + 3(1 + 3\gamma), \quad (3.30)$$

$$s = -\frac{3}{2}(1 + \gamma) \quad (3.31)$$

and we used

$$\frac{dV(\phi)}{d\phi} = H^2 \left(\frac{v'}{x} + v + \frac{2sv}{x} \right). \quad (3.32)$$

The equations (3.26) to (3.29) describe dynamical equations of a 4D phase space $\{x, y, z, v\}$. They are first order nonlinear differential equations and so their solutions may have chaotic behavior near possible critical points. If we want to seek stability of phase solutions of the above dynamical equations, then we must calculate their possible critical points for vacuum de Sitter era by setting $\gamma = -1$. Next we obtain eigenvalues of the corresponding Jacobi matrix and discuss their characteristics (see table 1).

3.2 Λ CDM de Sitter era

Inserting $\gamma = -1$ the dynamical equations (3.26), (3.27), (3.28), (3.29) can be rewritten as

$$x' = -(1 + 17\omega)x^2 - 3x - 5y - 8z + 4v + 108 \quad (3.33)$$

$$y' = -(5 + x)y, \quad (3.34)$$

$$z' = -(6 + x)z, \quad (3.35)$$

$$v' = \omega(43 + 102\omega)x^3 - 12\omega x^2 + 18(1 + 2\omega)x + 6(2 + 5\omega)xy \\ + 4(5 + 12\omega)xz - (13 + 24\omega)xv \quad (3.36)$$

where (3.30) and (3.31) take the following forms respectively.

$$q = 2\omega x^2 - 6, \quad (3.37)$$

and

$$s = 0. \quad (3.38)$$

For the vacuum de Sitter era, matter and radiation densities counterparts are negligible and so we must be set

$$y = 0, \quad z = 0. \quad (3.39)$$

x, v components of the critical points are determined by solving $x' = 0 = v'$. Inserting (3.39) and using (3.33) and (3.36) the equations $x' = 0$ and $v' = 0$ become

$$(1 + 17\omega)x_c^2 + 3x_c - (4v_c + 108) = 0 \quad (3.40)$$

and

$$\omega(43 + 102\omega)x_c^3 - 12\omega x_c^2 + 18(1 + 2\omega)x_c - (13 + 24\omega)x_c v_c = 0. \quad (3.41)$$

$(x_c = 0, v_c = -27)$ satisfies trivially the equations (3.40) and (3.41) for arbitrary values of ω and so it is one of de Sitter era critical points. If $x_c \neq 0$ then the equation (3.41) become

$$\omega(43 + 102\omega)x_c^2 - 12\omega x_c + 18(1 + 2\omega) - (13 + 24\omega)v_c = 0. \quad (3.42)$$

Eliminating v_c between (3.40) and (3.42), we obtain

$$(73\omega + 13)x^2 + (120\omega + 39)x - (2736\omega + 1476) = 0 \quad (3.43)$$

which has two solutions as

$$x_c^\pm = \frac{-3(40\omega + 13) \pm \sqrt{90368\omega^2 + 64736\omega + 8697}}{2(73\omega + 13)}. \quad (3.44)$$

Eliminating ω between (3.40) and (3.43) we obtain

$$v_c^\pm(x_c^\pm) = \frac{3(49x_c^4 + 167x_c^3 - 5892x_c^2 - 3528x_c + 49248)}{2(73x_c^2 + 120x_c - 2736)} \quad (3.45)$$

in which x_c must be inserted from (3.44). The solutions (3.44) and (3.45) show that there is two class of fixed points as

$$P_2^{de \ Sitter}(\omega) : (x_c^+(\omega), y_c = 0, z_c = 0, v_c^+(\omega)) \quad (3.46)$$

and

$$P_3^{de \ Sitter}(\omega) : (x_c^-(\omega), y_c = 0, z_c = 0, v_c^-(\omega)) \quad (3.47)$$

which make infinite number of critical points against different values of ω . Setting $x_c^+ = 0$ we obtain $\omega = -0.16856$ where $P_2^{de \ Sitter}$ reaches to $P_1^{de \ Sitter}$ and they become a unique fixed point. If we choose $x_c^- = 0$ we obtain $\omega = -0.56038$ where $P_3^{de \ Sitter}$ and $P_1^{de \ Sitter}$ become a unique fixed point. They have stable behavior for $\omega < 0$ and saddle (quasi-stable) for $\omega \geq 0$ (see figure 1 and table 1). Setting $x_c^+ = x_c^-$ we obtain $\omega = -0.17915$ where $P_{2,3}^{de \ Sitter}$ describes a quansi-stable state (see table 1). In general relativity approach of the BD theory itself we know $\omega \rightarrow +\infty$ where the BD scalar field reaches to a constant value. Hence we choose also samples $\omega = 40000$

and $\omega = -40000$ to obtain numerical values of critical points components as follows.

$$P_1^{deSitter}(\forall \omega \in \mathbb{R}) : (x_c = 0, y_c = 0, z_c = 0, v_c = -27), \quad (3.48)$$

$$P_{1,2}^{deSitter}(\omega = -0.16856) : (x_c = 0, y_c = 0, z_c = 0, v_c = -27), \quad (3.49)$$

$$P_3^{deSitter}(\omega = -0.16856) : (x_c = -27.02, y_c = 0, z_c = 0, v_c = 591.90), \quad (3.50)$$

$$P_{1,3}^{deSitter}(\omega = -0.56038) : (x_c = 0, y_c = 0, z_c = 0, v_c = -27), \quad (3.51)$$

$$P_2^{deSitter}(\omega = -0.56038) : (x_c = -1.013, y_c = 0, z_c = 0, v_c = -25.15), \quad (3.52)$$

$$P_{2,3}^{deSitter}(\omega = -0.17915) : (x_c = 112.92, y_c = 0, z_c = 0, v_c = 12953.62), \quad (3.53)$$

$$P_2^{deSitter}(\omega = 40000) : (x_c = 1.24, y_c = 0, z_c = 0, v_c = -21.99), \quad (3.54)$$

$$P_3^{deSitter}(\omega = 40000) : (x_c = -2.88, y_c = 0, z_c = 0, v_c = -5.99), \quad (3.55)$$

$$P_2^{deSitter}(\omega = -40000) : (x_c = -2.88, y_c = 0, z_c = 0, v_c = -5.99), \quad (3.56)$$

$$P_3^{deSitter}(\omega = -40000) : (x_c = 1.24, y_c = 0, z_c = 0, v_c = -21.99). \quad (3.57)$$

Other critical fixed points which can be considerable physically is for situations where at least one of roots of second order equations (3.40) and (3.42) have similar value (common root). To do so we must be set the following constraint condition between their coefficients.

$$\frac{(1 + 17\omega)}{\omega(43 + 102\omega)} = -\frac{3}{12\omega} = \frac{4v_c + 108}{(13 + 24\omega)v_c - 18(1 + 2\omega)} \quad (3.58)$$

leading to the following particular values.

$$\omega = \frac{47}{170} = 0.27647, \quad v_c = 68.870. \quad (3.59)$$

Inserting (3.59) the equations (3.40) and (3.43) read $x_c^+ = 7.9433$, $x_c^- = -8.4697$ and so we will have two other critical fixed points more as follows.

$$P_2^{deSitter}(\omega = 0.27647) : (x_c = 7.9433, y_c = 0, z_c = 0, v_c = 68.87) \quad (3.60)$$

and

$$P_3^{deSitter}(\omega = 0.27647) : (x_c = -8.4697, y_c = 0, z_c = 0, v_c = 68.87). \quad (3.61)$$

where nature of the fixed point (3.60) is stable but for (3.61) is unstable respectively (see table 1 and figure 1). Stability and/or instability of the above critical points can be follow via arrow diagrams of the dynamical equations (3.33) to (3.36) in figure 1 against different values of ω . In general, we can obtain time dependent solutions of the field equations of Λ CDM era for critical points $P_{1,2,3}^{de\ Sitter}(\omega)$ as follows.

$$P_{1,2,3}^{de\ Sitter} : \begin{pmatrix} \frac{\phi(t)}{\phi_0} = e^{x_c H t} \\ \rho_m = 0 \\ \rho_r = 0 \\ V(\phi) = v_c H^2 \phi \\ \zeta(t) = (2\omega x_c^2 - 6)\phi_0 H^2 e^{x_c H t} \\ \frac{a(t)}{a_0} = e^{H t} \end{pmatrix} \quad (3.62)$$

where H is Hubble constant which must be inserted via observational data and numerical values of (x_c, v_c) should be inserted from the equations (3.46) to (3.57) and/or (3.60) to (3.61). If we want to determine which of the above critical points have stable behavior then we must be calculate corresponding Jacobi matrix (1.4) and obtain eigenvalues as follows (see table 1).

$$J_{1,2,3}^{de\ Sitter}(\omega) = \begin{pmatrix} -3 - 2(1 + 17\omega)x_c & -5 & -8 & 4 \\ 0 & -(5 + x_c) & 0 & 0 \\ 0 & 0 & -(6 + x_c) & 0 \\ F(\omega, x_c, v_c) & 6(2 + 5\omega)x_c & 4(5 + 12\omega)x_c & -(13 + 24\omega)x_c \end{pmatrix} \quad (3.63)$$

where we defined

$$F(\omega, x_c, v_c) = 3\omega(43 + 102\omega)x_c^2 - 14\omega x_c + 18(1 + 2\omega) - (13 + 24\omega)v_c \quad (3.64)$$

and numerical values of ω, x_c, v_c should be inserted from the equations (3.48) to (3.57) and/or (3.60) to (3.61). We obtain corresponding secular equation as

$$\begin{aligned} &(\lambda + 5 + x_c)(\lambda + 6 + x_c)[\lambda^2 + [3 + (15 + 58\omega)x_c]\lambda \\ &+ (13 + 24\omega)[3x_c + 2(1 + 17\omega)x_c^2]] = 0 \end{aligned} \quad (3.65)$$

which has four eigenvalues as

$$\lambda_1 = -(5 + x_c), \quad \lambda_2 = -(6 + x_c),$$

$$\begin{aligned}
\lambda_3 &= -\frac{[3 + (15 + 58\omega)x_c]}{2} \\
&+ \frac{1}{2}\sqrt{[3 + (15 + 58\omega)x_c]^2 - 4(13 + 24\omega)x_c[3 + 2(1 + 17\omega)x_c]} \\
\lambda_4 &= -\frac{[3 + (15 + 58\omega)x_c]}{2} \\
&- \frac{1}{2}\sqrt{[3 + (15 + 58\omega)x_c]^2 - 4(13 + 24\omega)x_c[3 + 2(1 + 17\omega)x_c]} \quad (3.66)
\end{aligned}$$

where $\lambda_{1,2,3,4} < 0$ and $\lambda_{1,2,3,4} > 0$ describe stable and unstable state of the system. If some of the eigenvalues take positive values numerically but some other ones become negative then the system will take a quasi stable state namely saddle (see figure 1). We insert numerical values of (ω, x_c) from the equations (3.48) to (3.57) and/or (3.60) to (3.61) and collect numerical values of eigenvalues $\lambda_{1,2,3,4}^{de\ Sitter}$ in table 1 where first column in right side denotes their stability and/or instability nature. As a result of our work we now study experimental correspondence of our obtained solutions. Correspondence between Newton's gravity coupling parameter and the BD scalar field is well known as $\phi \equiv \frac{1}{G}$ from the BD gravity theory which by inserting (3.6) one infers [1]

$$\frac{1}{H} \left| \frac{\dot{G}}{G} \right|_{SBD} = \frac{1}{|1 + \omega|} \quad (3.67)$$

while for our model we will have

$$\frac{1}{H} \left| \frac{\dot{G}}{G} \right|_{VBD}^{\pm} = \left| \frac{-3(40\omega + 13) \pm \sqrt{90368\omega^2 + 64736\omega + 8697}}{2(73\omega + 13)} \right| \quad (3.68)$$

which in GR limits $\omega \rightarrow +\infty$ we can obtain nonzero counterpart of preferred reference frame effects as follows.

$$\lim_{\omega \rightarrow +\infty} \frac{1}{H} \left| \frac{\dot{G}}{G} \right|_{VBD}^{\pm} - \lim_{\omega \rightarrow +\infty} \frac{1}{H} \left| \frac{\dot{G}}{G} \right|_{SBD} \approx \begin{cases} 1.24; for + \\ 2.88; for - \end{cases} \quad (3.69)$$

where the present value of the Hubble constant is [1](see also [10,29])

$$H_{obs} = 7.24 \times 10^{-11} \text{ yr}^{-1}. \quad (3.70)$$

The above result predicts non-vanishing \dot{G} in presence of dynamical vector fields effects even in GR limits $\omega \gg 1$ which in BD gravity itself can not be detected. We now study dust era and its stability conditions of our model in the following subsection.

3.3 Dust era

For dust era matter density is non-vanishing $y \neq 0$ but for the radiation density we have $z = 0$ and corresponding barotropic index is $\gamma = 0$. Using the latter initial conditions the dynamical equations (3.26), (3.27), (3.28), (3.29) read

$$x' = -(1 + 17\omega)x^2 - 3x/2 - 5y + 4v + 108, \quad (3.71)$$

$$y' = -(2 + x)y, \quad (3.72)$$

$$z' = 0 \quad (3.73)$$

$$v' = \omega(43 + 102\omega)x^3 - 12\omega x^2 + 18(1 + 2\omega)x + 6(2 + 5\omega)xy - (13 + 24\omega)xv + 3v/2, \quad (3.74)$$

and (3.30) and (3.31) become respectively

$$q = 3 + 2\omega x^2 \quad (3.75)$$

and

$$s = -\frac{3}{2}. \quad (3.76)$$

Critical points are obtained by using (3.71) to (3.74) and setting $x' = 0 = y' = v'$ as

$$x_c = -2, \quad y_c(\omega) = \frac{10244\omega + 6173}{83}, \quad v_c(\omega) = \frac{14216\omega + 5496681}{83} \quad (3.77)$$

where $\omega > -0.6026$ because of positivity condition $y > 0$ of the matter density (3.22). However one can infer that ω dependent single critical point in the dust era become

$$P^{Dust} : \left(x_c = -2, y_c = \frac{10244\omega + 6173}{83}, z_c = 0, v_c = \frac{14216\omega + 5496681}{83} \right) \quad (3.78)$$

where (3.75) become

$$q_c = 3 + 8\omega. \quad (3.79)$$

Setting $\omega = \{-0.16856, -0.56038, -0.17915, 40000, 0.27647\}$ the above dust era critical point become respectively

$$P^{Dust}(\omega = -0.16856) : \left(x_c = -2, y_c = 53.57, z_c = 0, v_c = 67830.68 \right) \quad (3.80)$$

$$P^{Dust}(\omega = -0.56038) : \left(x_c = -2, y_c = 5.21, z_c = 0, v_c = 67761.90 \right), \quad (3.81)$$

$$P^{Dust}(\omega = -0.17915) : \left(x_c = -2, y_c = 52.26, z_c = 0, v_c = 67828.82 \right) \quad (3.82)$$

$$P^{Dust}(\omega = 40000) : \left(x_c = -2, y_c = 4.94 \times 10^6, z_c = 0, v_c = 7.09 \times 10^6 \right) \quad (3.83)$$

$$P^{Dust}(\omega = 0.27647) : \left(x_c = -2, y_c = 108.50, z_c = 0, v_c = 67908.78 \right). \quad (3.84)$$

One can calculate Jacobi matrix (1.4) for the critical point (3.78) as follows.

$$J^{Dust}(\omega) = \begin{pmatrix} \frac{(5+136\omega)}{2} & -5 & 0 & 4 \\ -\frac{(10244\omega+6173)}{83} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{(239592\omega^2+132055352\omega+71455359)}{83} & -12(2+5\omega) & 0 & \frac{(55+96\omega)}{2} \end{pmatrix} \quad (3.85)$$

where its secular equation defined by (1.5) become

$$\lambda[\lambda^3 - (30 + 116\omega)\lambda^2 + (\frac{1143185109}{332} + \frac{528335358\omega}{83} + \frac{1229280\omega^2}{83})\lambda + \frac{6173}{2} + 5122\omega] = 0. \quad (3.86)$$

Inserting $\omega = \{-0.16856, -0.56038, -0.17915, 40000, 0.27647\}$ we obtain numerical solutions of the eigenvalues equation (3.86) for critical points (3.80) to (3.84) and collect them into the table 1. Inserting (3.78) into the equations (3.20) to (3.25) and some simple integral calculations one can obtain dust era solutions as follows.

$$P^{Dust}(\omega) : \begin{pmatrix} \frac{\phi(t)}{\phi_0} = \left(\frac{t}{t_0}\right)^{\frac{4}{3}} \\ \rho_m(t) = \frac{(10244\omega+6173)}{747\pi} \frac{\phi_0}{t_0^2} \left(\frac{t}{t_0}\right)^{-\frac{2}{3}} \\ \rho_r(t) = 0 \\ V(\phi) = \left(\frac{16864\omega}{747} + \frac{7328908}{249}\right) \frac{\phi_0}{t_0^2} \left(\frac{\phi}{\phi_0}\right)^{-\frac{1}{2}} \\ \zeta(t) = \frac{4(3+8\omega)}{9} \frac{\phi_0}{t_0^2} \left(\frac{t}{t_0}\right)^{-\frac{2}{3}} \\ \frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \end{pmatrix} \quad (3.87)$$

where $\phi_0 = \phi(t_0)$, $a_0 = a(t_0)$, and t_0 is an arbitrary constant time. In the following subsection we study radiation era of the model and its stability conditions.

3.4 Radiation era

In case of radiation era, the matter density is vanishing $y = 0$ and barotropic index of state equation of radiation is $\gamma = \frac{1}{3}$. Inserting the latter initial conditions the dynamical equations (3.26), (3.27), (3.28), (3.29) read

$$x' = -(1 + 17\omega)x^2 - x - 8z + 4v + 108, \quad (3.88)$$

$$y' = 0, \quad (3.89)$$

$$z' = -(2 + x)z \quad (3.90)$$

$$v' = \omega(43 + 102\omega)x^3 - 12\omega x^2 + 18(1 + 2\omega)x + 4(5 + 12\omega)zx - (13 + 24\omega)xv + 2v, \quad (3.91)$$

where (3.30) and (3.31) become respectively

$$q = 6 + 2\omega x^2 \quad (3.92)$$

and

$$s = -2. \quad (3.93)$$

Critical points are obtained by using (3.88) to (3.91) and setting $x' = 0 = z' = v'$ as

$$x_c = -2, \quad v_c = \frac{283}{4} + \frac{349\omega}{2}, \quad z_c = \frac{315\omega}{4} + \frac{389}{8} \quad (3.94)$$

where positivity condition of the radiation density (3.23) restricts us to choose $z > 0$ and so $\omega > -0.59846$. Thus critical point in the radiation era become

$$P^{Radiation} : (x_c = -2, y_c = 0, z_c = \frac{315\omega}{4} + \frac{389}{8}, v_c = \frac{283}{4} + \frac{349\omega}{2}) \quad (3.95)$$

where

$$q_c = 6 + 8\omega, \quad s_c = -2. \quad (3.96)$$

Setting $\omega = \{-0.16856, -0.56038, -0.17915, 40000, 0.27647\}$ the above radiation era critical point become respectively

$$P^{Rad}(\omega = -0.16856) : \left(x_c = -2, y_c = 0, z_c = 35.35, v_c = 41.34 \right) \quad (3.97)$$

$$P^{Rad}(\omega = -0.56038) : \left(x_c = -2, y_c = 0, z_c = 4.50, v_c = -27.04 \right), \quad (3.98)$$

$$P^{Rad}(\omega = -0.17915) : \left(x_c = -2, y_c = 0, z_c = 34.52, v_c = 39.49 \right) \quad (3.99)$$

$$P^{Rad}(\omega = 40000) : \left(x_c = -2, y_c = 0, z_c = 3.15 \times 10^6, v_c = 6.98 \times 10^6 \right) \quad (3.100)$$

$$P^{Rad}(\omega = 0.27647) : \left(x_c = -2, y_c = 0, z_c = 70.40, v_c = 118.99 \right). \quad (3.101)$$

One can calculate Jacobi matrix (1.4) for the radiation era critical point (3.95) as follows.

$$J^{radiation}(\omega) = \begin{pmatrix} 3 + 68\omega & 0 & -8 & 4 \\ 0 & 0 & 0 & 0 \\ -\left(\frac{315\omega}{4} + \frac{389}{8}\right) & 0 & 0 & 0 \\ -1632\omega^2 - \frac{979\omega}{2} + \frac{283}{4} & 0 & -8(5 + 12\omega) & 4(7 + 12\omega) \end{pmatrix} \quad (3.102)$$

where its secular equation (1.5) become

$$\lambda[\lambda^3 - (116\omega + 31)\lambda^2 + (9792\omega^2 + 3376\omega - 588)\lambda + 5040\omega + 3112] = 0. \quad (3.103)$$

Inserting $\omega = \{-0.16856, -0.56038, -0.17915, 40000, 0.27647\}$ we solve (3.103) and obtain numerical values of eigenvalues for critical points (3.97) to (3.101) and collect them into the table 1. Inserting (3.95) and (3.96) into the equations (3.20) to (3.25) and some simple integral calculations one finds

$$P^{Rad}(\omega) : \begin{pmatrix} \frac{\phi(t)}{\phi_0} = \frac{t}{t_0} \\ \rho_m(t) = 0 \\ \rho_r(t) = \frac{1}{128\pi} \left(315\omega + \frac{389}{2} \right) \frac{\phi_0}{t_0^2} \left(\frac{t}{t_0} \right)^{-1} \\ V(\phi) = \left(\frac{349\omega}{8} + \frac{283}{16} \right) \frac{\phi_0}{t_0^2} \left(\frac{\phi}{\phi_0} \right)^{-1} \\ \zeta(t) = \frac{(3+4\omega)}{2} \frac{\phi_0}{t_0^2} \left(\frac{t}{t_0} \right)^{-1} \\ \frac{a(t)}{a_0} = \left(\frac{t}{t_0} \right)^{\frac{1}{2}} \end{pmatrix} \quad (3.104)$$

where $\phi_0 = \phi(t_0)$, $a_0 = a(t_0)$, and t_0 is an arbitrary constant time.

<i>Fixed point</i>	<i>Eigenevalues : ($\lambda_1, \lambda_2, \lambda_3, \lambda_4$)</i>	<i>Nature</i>
$P_{1;\forall\omega\in\mathbb{R}}^{deS}$	(-5, -6, 0, -6)	<i>stable</i>
$P_{1,2;\omega_1=-0.16856}^{deS}$	(-5, -6, 0, -6)	<i>stable</i>
$P_{3;\omega=-0.16856}^{deS}$	(22.02, 21.02, 483.90, -207.63)	<i>saddle</i>
$P_{1,3;\omega=-0.56038}^{deS}$	(-5, -6, 0, -6)	<i>stable</i>
$P_{2;\omega=-0.56038}^{deS}$	(-3.99, -4.99, -0.91, -40.55)	<i>stable</i>
$P_{2,3;\omega=-0.17915}^{deS}$	(-117.92, -118.90, 917.93, -1964.90)	<i>saddle</i>
$P_{2;\omega=40000}^{deS}$	(-6.24, -7.24, -2.38×10^6 , -3.37×10^6)	<i>stable</i>
$P_{3;\omega=40000}^{deS}$	(-2.12, -3.12, 7.83×10^6 , 5.53×10^6)	<i>saddle</i>
$P_{2;\omega=-40000}^{deS}$	(-2.12, -3.12, -5.53×10^6 , -7.83×10^6)	<i>stable</i>
$P_{3;\omega=-40000}^{deS}$	(-6.24, -7.24, 3.37×10^6 , 2.38×10^6)	<i>saddle</i>
$P_{2;\omega=0.27647}^{deS}$	(-12.94, -13.94, -187.11, -311.44)	<i>stable</i>
$P_{3;\omega=0.27647}^{deS}$	(3.47, 2.37, 332.61, 187.11)	<i>untable</i>
$P_{\omega=0.27647}^{Dus}$	(0, 31.04+2281.09i, -0.0009, 31.04-2281.09i)	<i>saddle</i>
$P_{\omega=40000}^{Dus}$	(0, 100, $(2.3 - 4.3i) \times 10^6$, $(2.3 + 4.3i) \times 10^6$)	<i>unstable</i>
$P_{\omega=-0.17915}^{Dus}$	(0, 4.61+1517.70i, -0.0009, 4.61-1517.70i)	<i>saddle</i>
$P_{\omega=-0.56038}^{Dus}$	(0, 0.002, 328.07, -363.07)	<i>saddle</i>
$P_{\omega=-0.16856}^{Dus}$	(0, 5.22+1539.73i, -0.0009, 5.22-1539.73i)	<i>saddle</i>
$P_{\omega=0.27647}^{Rad}$	(0, 32.24+14.69i, -3.41, 32.24-14.69i)	<i>saddle</i>
$P_{\omega=40000}^{Rad}$	(0, 0, $(2.32 - 3.21i) \times 10^6$, $(2.32 + 3.21i) \times 10^6$)	<i>unstable</i>
$P_{\omega=-0.17915}^{Rad}$	(0, 2.46, 34.09, -26.33)	<i>saddle</i>
$P_{\omega=-0.56038}^{Rad}$	(0, -16.75+17.26i, -0.50, -16.75-17.26i)	<i>stable</i>
$P_{\omega=-0.16856}^{Rad}$	(0, 2.51, 34.82, -25.88)	<i>saddle</i>

Table 1: Numerical values of eigenvalues for Λ CDM de Sitter, dust and radiation eras where the corresponding space time scale factor become $a_{deS}(t) \sim e^{Ht}$, $a_{Dus}(t) \sim t^{\frac{2}{3}}$ and $a_{Rad}(t) \sim t^{\frac{1}{2}}$ respectively.

4 Concluding remark

Applying VBD gravity [3] in presence of additional perfect fluid matter and self interaction potential action functionals we studied flat FRW space time dynamics. We applied dynamical system approach to seek stable critical points for vacuum de Sitter, dust and radiation eras. To do so we calculate eigenvalues of the corresponding Jacobi matrix defined on 4D phase space. In general, we obtain 3 type critical fixed points for de Sitter era but 1 type

for dust and radiation eras. Nature of these critical points are depended to choose numerical values of the BD parameter ω . When the potential behaves as (effective cosmological) constant then one of the critical fixed point in de Sitter era become stable for $\omega < 0$ and saddle for $\omega \geq 0$. While for linear potential $V(\phi) \sim \phi$ (variable cosmological parameter) there is still a stable critical point in de Sitter era but for particular value of $\omega = 0.27647$. There is not obtained conditions where the all 3 fixed points reach to a unique critical fixed point. While for $\omega = \{-0.16856, -0.56038, -0.17915\}$ there is at least 2 out of 3 critical fixed points in de Sitter era which become unique (see table 1). In dust era the system become stable for $\omega = -0.56038$ but behaves as unstable by vanishing matter density for $\omega = 0.27647$ (see figure 1). The latter case predicts a phase transition from matter to vacuum de Sitter era. Radiation era become quasi-stable for $\omega = \{-0.56038, 0.27647\}$ by vanishing the radiation density. This result predicts a phase transition between radiation and dust eras for particular value of $\omega = 0.27647$. Comparing diagrams given in figure 1 we can understand $\omega_{unique} = 0.27647$ is important value for the BD parameter in the used gravity model [3] where flat FRW space time tolerates a radiation era by supporting potential $V(\phi) \sim \phi^{-1}$, then transmit to a dust era by supporting a potential as $V(\phi) \sim \phi^{-\frac{1}{2}}$ and finally transmit to a vacuum de Sitter era by supporting a linear potential $V(\phi) \sim \phi$. As a result of our work we consider time dependent fluctuations of Newton's coupling parameter $\dot{G}(t)$ obtained from BD gravity itself and compare it with our results in GR limits $\omega \rightarrow 40000$. Non-vanishing counterparts denotes to preferred reference frame effects coming from the used alternative model in this work. As extensions of our work we seek preferred reference frame effects [3] on anisotropy of Bianchi's cosmology and galaxy rotation curves too in our next work.

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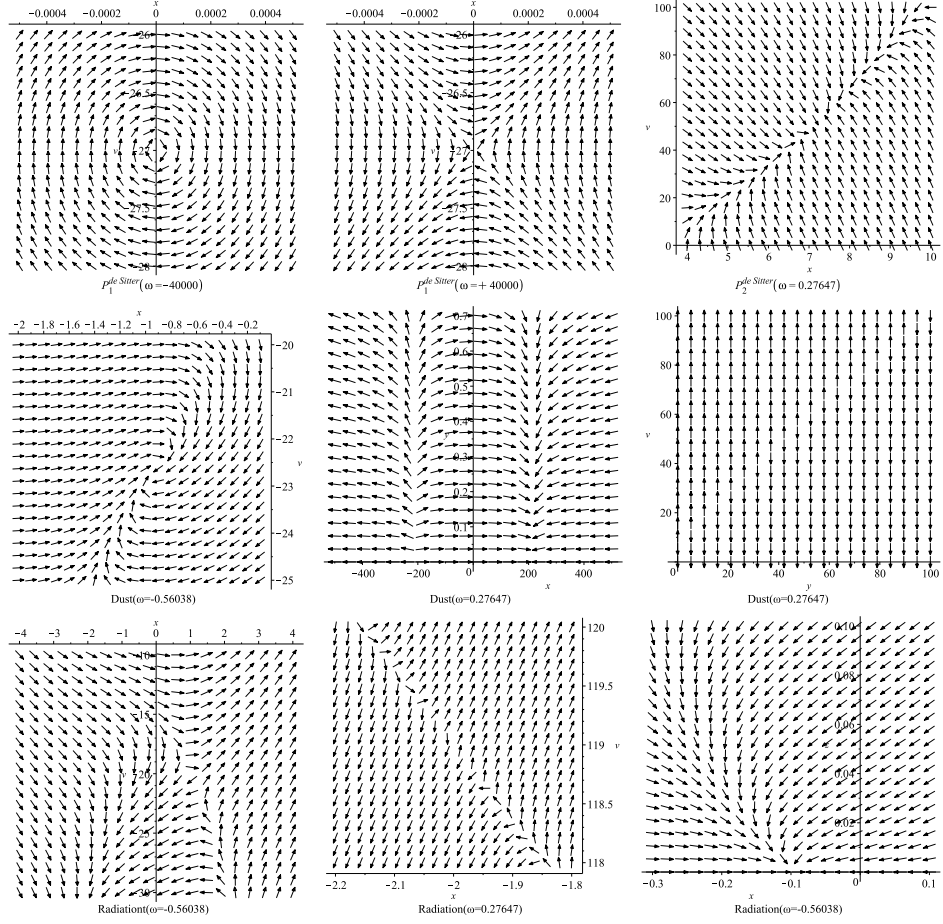


Figure 1: Arrow diagrams of critical fixed points for de Sitter, dust and radiation eras